

Quantization and Asymptotic Behaviour of ε_{V^k} Quantum Random Walk on Integers*

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Abstract

Quantization and asymptotic behaviour of a variant of discrete random walk on integers are investigated. This variant, the ε_{V^k} walk, has the novel feature that it uses many identical quantum coins keeping at the same time characteristic quantum features like the quadratically faster than the classical spreading rate, and unexpected distribution cutoffs. A weak limit of the position probability distribution (pd) is obtained, and universal properties of this arch sine asymptotic distribution function are examined. Questions of driving the walk are investigated by means of a quantum optical interaction model that reveals robustness of quantum features of walker's asymptotic pd, against stimulated and spontaneous quantum noise on the coin system.

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I. THE ε_{V^k} QUANTUM RANDOM WALK ON \mathbb{Z}

Recent research activity on the topic of the so called quantum random walks has achieved a number of interesting results and extensions of the usual notion of random walk (see e.g the following articles and reviews [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). Our present study investigates further quantum walks and their asymptotic statistical behaviour. Let us consider a walker system with Hilbert space $H_w = \text{span}\{|n\rangle \mid n \in \mathbb{Z}\}$, and a coin system with space $H_c = \text{span}\{|+\rangle, |-\rangle\}$. Let the evolution operator be of the form

$$V = (P_+ \otimes E_+ + P_- \otimes E_-)U \otimes \mathbf{1}. \quad (1)$$

This acts on the tensor product space $H_c \otimes H_w$. Here U is a unitary operator acting on the coin Hilbert space and P_+, P_- are the projection operators in the distinguished basis $\{|\pm\rangle\}$ of H_c . Also E_\pm are the right/left one step operators in the distinguished basis $\{|n\rangle \mid n \in \mathbb{Z}\}$ of H_w . As they are commuting they share the eigenbasis $\left\{|\phi\rangle = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\phi} |n\rangle \mid \phi \in [0, 2\pi]\right\}$, of orthogonal elements viz. $\langle \phi | \phi' \rangle = \frac{1}{2\pi} \delta(\phi - \phi')$, which satisfy the eigenvalue equations $E_\pm |\phi\rangle = e^{\pm i\phi} |\phi\rangle$.

The evolution operator V acts on the state $|\phi\rangle$ to give $V |\phi\rangle = V(\phi) |\phi\rangle$, where $V(\phi) = (e^{i\phi} P_+ + e^{-i\phi} P_-)U = e^{i\phi\sigma_3}U$. Here $V(\phi)$ is a unitary operator that acts on H_c . Suppose now that we consider the ε_{V^k} model [14, 15], in which the one-step evolution of the walker density matrix ρ^0 is given by $\rho^1 = \varepsilon_{V^k}(\rho^0) \equiv \text{Tr}_{H_c}(V^k \rho_c \otimes \rho^0 V^{\dagger k})$, where ρ_c is the coin density matrix. Here ε_{V^k} is a completely positive trace preserving (CPTP) map that acts on $H_w \otimes H_w^*$, however, since it has the property that it maps density matrices to density matrices we can reduce its action to the convex subset of density matrices $D(H_w) \subset H_w \otimes H_w^*$. Suppose that $\rho^0 = \int_0^{2\pi} \int_0^{2\pi} \rho(\phi, \phi') |\phi\rangle \langle \phi'|$. We have that $\varepsilon_{V^k}(|\phi\rangle \langle \phi'|) = \text{Tr}_{H_c}(V^{\dagger k}(\phi') V^k(\phi) \rho_c) |\phi\rangle \langle \phi'| \equiv A(\phi, \phi') |\phi\rangle \langle \phi'|$, hence $\rho^1 = \int_0^{2\pi} \int_0^{2\pi} \rho(\phi, \phi') A(\phi, \phi') |\phi\rangle \langle \phi'|$. The n -step evolved walker density matrix is $\rho^n = \int_0^{2\pi} \int_0^{2\pi} \rho(\phi, \phi') A(\phi, \phi')^n |\phi\rangle \langle \phi'|$, where $A(\phi, \phi')$ is the *characteristic function* of the quantum random walk.

The *position observable* $L|m\rangle = m|m\rangle$, $m \in \mathbb{Z}$, and its positive powers lend themselves to study the statistical moments of the quantum walker after n steps. To emphasize the connection with the classical walk we also introduce a sequence of classical random variables $L_c^{(n)}$ over the common sample space \mathbb{Z} , that correspond to the positions of the classical walker

after n steps, with probability distribution $P(L_c^{(n)} = m) = \text{Tr}(|m\rangle\langle m|\rho^n) = \langle m|\rho^n|m\rangle$. Then we obtain for the statistical moments

$$\langle L^s \rangle_n \equiv \text{Tr}(L^s \rho^n) = \frac{1}{2\pi i^s} \int_0^{2\pi} d\phi [\partial_\phi^s [\rho(\phi, \phi') A^n(\phi, \phi')]]_{\phi'=\phi} = \sum_{m \in \mathbb{Z}} m^s P(L^{(n)} = m) \equiv \langle L_c^s \rangle_n \quad (2)$$

A further study of the asymptotic behavior of the first and second moment leads to the following

Lemma 1 *The mean position of the quantum random walk is of the form $\mu_n = \langle L \rangle_n = K_1 n + \mu_0$, while the variance is of the form $\sigma_n^2 = \langle L^2 \rangle_n - \langle L \rangle_n^2 = K_2 n^2 + K_3 n + \sigma_0^2$, where K_1, K_2, K_3 depend on the initial coin density matrix, the initial walker density matrix and the tracing scheme, however they are independent of the number of evolution steps taken, and μ_0 and σ_0^2 are the mean and variance of the initial position distribution. In particular, if the quantum walker is initially in the at the position 0, that is $\rho_0 = |0\rangle\langle 0|$, then $\mu_n = K_1 n$ and $\sigma_n^2 = K_2 n^2 + K_3 n$.*

Remark on U -quantization: Let us consider the general QRW as introduced above, in the particular case where $U = \mathbf{1}$, we have that $\varepsilon_{V^k}(\rho^0) = \rho_{c++} E_+^k \rho^0 E_+^{\dagger k} + \rho_{c-} E_-^k \rho^0 E_-^{\dagger k}$. This admits the interpretation that the walker shifts its position k steps to the right (left) with probability ρ_{c++} (ρ_{c-}). So if we are given a classical random walk on \mathbb{Z} with nearest neighbor transition probabilities $p, 1-p$, we can select a coin density matrix whose diagonal elements are these transition probabilities in the preferred coin basis, and view in this way the classical random walk as an ε_{V^k} quantum random walk with $U = \mathbf{1}$, and step size k . A natural way then to quantize this classical walk is a unitary choice so that $U \neq \mathbf{1}$. This amounts to a continuous deformation of classical walk that may result into a new walk with novel quantum features, as its known. Also since U acts only on coin space, and we also need to insert a consistent coin density matrix, this U -quantization procedure as may be called, should be understood as a quantization of the coin of a classical random walk. Generally the so obtained U -quantized random walks may admit solutions that employ off diagonal walker density matrices, and these are considered to be genuine quantum mechanical random walks. In fact the U -quantization is not restricted to walks on integers, but it can be used more generally to quantized classical walks on other fields of numbers and general lattice systems.

II. ASYMPTOTIC BEHAVIOUR OF QUANTUM RANDOM WALK

Next we study the behaviour of the walker when the number of steps n , is large. In this case we have that

$$\langle L^s \rangle_n = \frac{n^s}{2\pi i^s} \int_0^{2\pi} d\phi \rho(\phi, \phi) [\partial_\phi A(\phi, \phi')]_{\phi'=\phi}^s + O(n^{s-1}) \equiv \frac{n^s}{2\pi} \int_0^{2\pi} d\phi \rho(\phi, \phi) h(\phi)^s + O(n^{s-1}). \quad (3)$$

Here we have defined the *asymptotic characteristic function* $h(\phi)$ of the walk as

$$h(\phi) = -i [\partial_\phi A(\phi, \phi')]_{\phi'=\phi} = \text{ImTr}_{H_c}(V^{k\dagger}(\phi) (V^k(\phi))' \rho_c). \quad (4)$$

For the ε_{V^k} model it reads

$$h(\phi) = \text{Tr}_{H_c} [(\sigma + V^\dagger(\phi)\sigma V(\phi) + \dots + V^{\dagger k-1}(\phi)\sigma V^{k-1}(\phi))\rho_c] \quad (5)$$

where $\sigma = U^\dagger \sigma_3 U$ is a rotated σ_3 Pauli matrix.

As the following theorem states the sequence of probability measures $\{\frac{1}{n}\text{Tr}(|m\rangle\langle m|\rho^n) = P(\frac{L_c^{(n)}}{n} = \frac{m}{n}), n = 1, 2, \dots\}$ has all its moments converging to the moments of $Y = h(\phi)$, as n goes to infinity, where ϕ stands for a random variable (rv) on the circle with measure $\frac{1}{2\pi}\rho(\phi, \phi)$.

Theorem 2 *The sequence of classical random variables $\frac{L_c^{(n)}}{n}, n = 1, 2, \dots$, corresponding to the sequence of quantum observables, converges weakly to the random variable $Y = h(\phi)$, where ϕ is now a random variable with values on the circle and measure $\frac{1}{2\pi}\rho(\phi, \phi)$. The probability distribution function for Y is given by the formula*

$$P(y_1 \leq Y \leq y_2) = \frac{1}{2\pi} \int_{y_1 \leq h(\phi) \leq y_2} \rho(\phi, \phi) d\phi = \frac{1}{2\pi} \sum_i \int_{y_1}^{y_2} \rho(h_i^{-1}(y), h_i^{-1}(y)) \frac{1}{|h'(h_i^{-1}(y))|} dy, \quad (6)$$

where it is assumed that locally in the interval $[y_1, y_2]$ the function h admits a number of local inverses, labelled by i .

Remarks: *i)* The dual $\varepsilon^*(X)$ of a CPTP $\varepsilon(\rho) = \sum_i A_i \rho A_i^\dagger$ operating on a observable X and a density operator ρ respectively is obtained to be $\varepsilon^*(X) = \sum_i A_i^\dagger X A_i$ *ii)* The asymptotic characteristic function has a finite Fourier expansion, hence it is locally invertible except from a set of measure zero, if the function is not a constant. Further the inverse is differentiable

except from a set of measure zero. Hence the asymptotic distribution of Y given above holds as long as h is not a constant. A somewhat similar theorem is obtained in [16] for the original QRW with two essential differences. First, in [16] the sample space of rv ϕ was enlarged by taking a product of the circle with a set of two points. Second, the initial walker-coin state information was encoded in the measure while in the present case only the initial walker state information is encoded in the corresponding measure.

It is now reasonable to ask what happens if h is constant function. This is answered by the following lemma:

Lemma 3 *If the asymptotic characteristic function h is constant then $\sigma_n^2 = \langle L^2 \rangle_n - \langle L \rangle_n^2 = K_3 n + \sigma_0^2$, which means the quantum random walk spreads classically.*

This can be proven simply by considering the asymptotic expansion of the moments. As an example, let $|+\rangle$ be the initial coin state, and choose $U = e^{i\frac{\pi}{4}\sigma_2}$. The walker state is initially $|0\rangle$, while the model is taken to be the ε_{V^2} . In this case it turns out that $h(\phi) = -\cos(2\phi)$, while $\rho(\phi, \phi) = 1$.

Remarks: *i)* The range of values of $h(\phi)$ is the interval $[-1, 1]$. Given that the possible values of walker position after n steps extends on the interval $[-2n, 2n]$, the normalized position asymptotic range of values is expected to be the interval $[-2, 2]$. This means that the asymptotic position distribution experiences a sudden cutoff at half the spread one would expect. This cutoff has also been observed with different quantum evolution schemes [8], [18], [9], [16]. *ii)* Since ϕ takes values in the interval $[0, 2\pi]$, there are four relevant inverses of h with domain the interval $[-1, 1]$, and range in $[0, 2\pi]$. All these inverses satisfy the relation $h'(h_i^{-1}(y)) = \pm 2\sqrt{1 - y^2}$. The resulting asymptotic distribution is

$$P(y_1 \leq Y \leq y_2) = \frac{1}{\pi} \int_{y_1}^{y_2} \frac{1}{\sqrt{1 - y^2}} dy. \quad (7)$$

It is worth mentioning that this is the same distribution as the one obtained in [19], in a different context.

In fact a more general result about the asymptotic distribution can be proved along the same lines as the following theorem states:

Theorem 4 *Whatever the initial coin density matrix and whatever the unitary coin reshuffling matrix for the ε_{V^2} quantum random walk, if the asymptotic characteristic function $h(\phi)$*

is not constant then the normalized random variable $Y' = \frac{Y-\mu}{\sqrt{2}\sigma}$ is distributed according to the distribution of eq. (7).

This theorem is to be understood as a quantum ε_{V^2} version of the De Moivre-Laplace theorem for the normal approximation of the binomial distribution. In fact the corresponding theorem one would obtain, if the quantum system followed the ε_V evolution, would be precisely the De Moivre-Laplace theorem. In the case of the $\varepsilon_{V^{n+1}}$ QRW the asymptotic characteristic function is of the form $h(\phi) = \mu - \sum_{m=1}^n A_m \cos 2m(\phi + \alpha_m)$, where A_m and α_m depend on the coin state and the reshuffling matrix U . If we define $Y_m = \cos 2m(\phi + \alpha_m)$ then it is not difficult to show that it is distributed according to eq. (7) for any m . Hence we obtain the following proposition:

Proposition 5 *If the asymptotic characteristic function is not constant then the normalized asymptotic position random variable Y is written as $Y = \mu + \sum_{m=1}^n A_m Y_m$, where each Y_m is distributed according to eq. (7), with $\sigma_Y^2 = \frac{1}{2} \sum_{m=1}^n A_m^2$. (It should be noted however that the rv's Y_m are strongly correlated).*

III. CAVITY DRIVEN QUANTUM RANDOM WALK

To probe the effect of initial coin state upon the long time behaviour of quantum walker system we introduce an operational way to exercise control on that coin state. If the coin, which is taken to be a two level atom, is prepared through its interaction with a single electromagnetic (EM) mode optical cavity on resonance and if the coin-cavity interaction is assumed to be of Jaynes-Cummings model (JCM), (see [15], and references therein, also for the case of more general models), then its dynamic state is described as

$$\varepsilon_U(\rho_C) = Tr_f U(t)(\rho_C \otimes \rho_f)U(t)^\dagger = S_0 \rho_C S_0^\dagger + S_1 \rho_C S_1^\dagger. \quad (8)$$

The CPTP map ε_U , describes a stimulate transition of the coin quantum system due to its interaction with the cavity mode, and is determined by the unitary evolution $U(t)$ of JCM, and the initial field state which is taken to be a pure state $\rho_f = |f\rangle\langle f|$, where f stands for the cavity photon number. The state of atomic coin system after crossing the cavity is described by the reduced density matrix given above, where t stands for the crossing time.

The general case of an initial sharp number state $|f\rangle = |r\rangle$, $r = 0, 1, 2, \dots$, leads to the reduced coin density matrix

$$\varepsilon_U(\rho_C) = A_1 \rho_C A_1^\dagger + A_2 \rho_C A_2^\dagger + A_3 \rho_C A_3^\dagger \quad (9)$$

with generators

$$\begin{aligned} A_1 &= \begin{pmatrix} \cos(\lambda t \sqrt{r+1}) & 0 \\ 0 & \cos(\lambda t \sqrt{r}) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ \sin(\lambda t \sqrt{r+1}) & 0 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & \sin(\lambda t \sqrt{r}) \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (10)$$

The trace preservation of this map requires that $A_1^\dagger A_1 + A_2^\dagger A_2 + A_3^\dagger A_3 = \mathbf{1}$.

Consider the special case $\rho_C = |c\rangle \langle c|$ where $|c\rangle = \cos \chi |+\rangle + i \sin \chi |-\rangle$, that leads to symmetric walk about the origin. After the coin crosses the JCM cavity, its state becomes $\varepsilon_U(\rho_C)$, where

$$\begin{aligned} \varepsilon_U(\rho_C) &= \frac{1}{2} \mathbf{1} + \frac{1}{2} \sin(2\chi) \cos(\lambda \sqrt{r+1} t) \cos(\lambda \sqrt{r} t) \sigma_2 \\ &\quad + \frac{1}{2} \left[\cos(2\lambda \sqrt{r+1} t) \cos^2 \chi - \cos(2\lambda \sqrt{r} t) \sin^2 \chi \right] \sigma_3. \end{aligned} \quad (11)$$

Suppose now the prepared atomic coin is used to drive a quantum random walk according to the ε_{V^2} model for $U = e^{i\frac{\pi}{4}\sigma_2}$. In this case the asymptotic characteristic function reads

$$\begin{aligned} h(\phi; \chi, t) &= \left[-\cos(2\lambda t \sqrt{r+1}) \cos^2 \chi + \cos(2\lambda t \sqrt{r}) \sin^2 \chi \right] \cos(2\phi) \\ &\quad + \left[\sin(2\chi) \cos(\lambda t \sqrt{r+1}) \cos(\lambda t \sqrt{r}) \right] \sin(2\phi). \end{aligned} \quad (12)$$

To evaluate the limit probability distribution function, we rewrite the last equation as $h(\phi; \chi, t) = C(\chi, t) \cos[2\phi - \Lambda(\chi, t)]$, where we have introduced the functions $A(\chi, t) = -\cos(2\lambda t \sqrt{r+1}) \cos^2 \chi + \cos(2\lambda t \sqrt{r}) \sin^2 \chi$, $B(\chi, t) = \sin(2\chi) \cos(\lambda t \sqrt{r+1}) \cos(\lambda t \sqrt{r})$, from which we define the two functions $C(\chi, t) = \sqrt{A(\chi, t)^2 + B(\chi, t)^2}$ and $\tan \Lambda(t) = \frac{B(\chi, t)}{A(\chi, t)}$. If we define the rv $Y = h(\phi; t)$, then the normalized asymptotic walker position distribution becomes

$$P(y; \chi, t) = \frac{1}{\pi \sqrt{C(\chi, t)^2 - y^2}}, \quad -1 \leq y \leq 1. \quad (13)$$

This pdf depends on the crossing time t and on the initial coin state through χ . The mean and standard deviation derived from the above pdf are respectively

$$\mu = \lim_{n \rightarrow \infty} \langle \frac{L}{n} \rangle_n = \int_0^{2\pi} h(\phi; \chi, t) \frac{d\phi}{2\pi} = 0, \quad (14)$$

$$\sigma(\chi, t)^2 = \lim_{n \rightarrow \infty} \langle \left(\frac{L}{n} \right)^2 \rangle_n = \int_0^{2\pi} h(\phi; \chi, t)^2 \frac{d\phi}{2\pi} = \frac{C(\chi, t)^2}{2}. \quad (15)$$

It is useful to observe at this stage[15], that the distribution of eq. (13), is robust under changes in χ, t , unless it happens that $C(\chi, t) = 0$. In this case the distribution collapses. Inspection of the functions $A(\chi, t)$, $B(\chi, t)$ and $C(\chi, t)$ given above reveals that if $\chi = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, namely if $|c\rangle = \{|+\rangle, i|-\rangle, -|+\rangle, -i|-\rangle\}$ respectively, and $t = \{\frac{(2k+1)\pi}{4\lambda\sqrt{r+1}}, \frac{(2k+1)\pi}{4\lambda\sqrt{r}}, \frac{(2k+1)\pi}{4\lambda\sqrt{r+1}}, \frac{(2k+1)\pi}{4\lambda\sqrt{r}}\}$, $k \in \mathbb{Z}$, then $C(\chi, t) = 0$. The specific relations among the four coin states and interaction times as given above result into only two different pairs of coin density matrices and interaction times namely, $\rho_C = |+\rangle\langle+|$, $t = \frac{(2k+1)\pi}{4\lambda\sqrt{r+1}}$, and $\rho_C = |-\rangle\langle-|$, $t = \frac{(2k+1)\pi}{4\lambda\sqrt{r}}$, for which $C(\chi, t) = 0$. If either of these two conditions occur we say that a *resonance condition* takes place between the field and the two level atom. In this case, $\langle L^2 \rangle_n \sim n$, so we loose the quadratic diffusion time speed up, characterizing the quantum random walk. In such a case the asymptotic behaviour of the standard deviation agrees with that of a classical random walk. More precisely what happens is that in all the above cases, the exited coin is in the maximally mixed state $\varepsilon_U(\rho_C) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}\mathbf{1}$. If this coin system is used in V^2 QRW, then the final one-step density matrix for the walker system becomes

$$\rho_W \rightarrow \varepsilon_{V^2}(\rho_W) = \frac{1}{2}\rho_W + \frac{1}{4}E_+^2\rho_W E_+^{\dagger 2} + \frac{1}{4}E_-^2\rho_W E_-^{\dagger 2}. \quad (16)$$

Due to the last equation, if ρ_W is diagonal initially then so is finally. Hence we really have a classical one-step transition that leads to Gaussian statistics for large n , once we normalize L to $\frac{L}{\sqrt{n}}$. That implies that on resonance the walk becomes fully classical. This analysis makes obvious the fact that a judicious choice of the initial coin state permits us to tune the interaction time in a Jaynes-Cummings cavity, so that we have a quantum to classical transition on the asymptotic behavior of the ε_{V^2} quantum random walk. This conclusion makes the quantum optical experimental investigation of this idea worthwhile.

IV. SPONTANEOUS EMISSION IN COIN SYSTEM

What we are going to study in this section is the effect of a spontaneously emitting coin system on the QRW evolution. We will assume that the coins that come into contact with the walker system are corrupted by spontaneous emission from the state $|+\rangle$ to the state $|-\rangle$. The state $|-\rangle$ is assumed to be decay stable, and state $|+\rangle$ is metastable and has probability γ of decaying to the state $|-\rangle$. Let the initial coin state be $\rho_C^0 = p_0 |-\rangle \langle -| + p_1 |+\rangle \langle +|$, then the effect of spontaneous emission is to modify this density matrix to $\rho_C^1 = (p_0 + \gamma p_1) |-\rangle \langle -| + (1 - \gamma)p_1 |+\rangle \langle +|$. This effect can be captured by the generators[17],

$$\begin{aligned} T_0 |-\rangle &= |-\rangle & T_1 |-\rangle &= 0 \\ T_0 |+\rangle &= \sqrt{1-\gamma} |+\rangle & T_1 |+\rangle &= \sqrt{\gamma} |-\rangle \end{aligned} \quad (17)$$

In terms of these generators the ε_{SE} CPTP map that transforms ρ_C^0 to ρ_C^1 is

$$\rho_C^1 = \varepsilon_{SE}(\rho_C^0) = T_0 \rho_C^0 T_0^\dagger + T_1 \rho_C^0 T_1^\dagger. \quad (18)$$

Of course $T_0^\dagger T_0 + T_1^\dagger T_1 = \mathbf{1}$. In fact such a spontaneous de-excitation can happen as result of a stimulated transition as e.g in the case where the coin passes through an initially empty quantum optical cavity, (see the case of JCM previously), in which case $\gamma = \sin^2 \lambda t$, and $T_0 = S_0(t)$, $T_1 = S_1(t)$, where t stands for the time spent by the coin inside the empty cavity. Indeed for the particular initial vacuum state $|f\rangle = |0\rangle$, we obtain the map

$$\rho_C^1 = \varepsilon_U(\rho_C^0) = \varepsilon_{SE}(\rho_C^0) = S_0 \rho_C S_0^\dagger + S_1 \rho_C S_1^\dagger \quad (19)$$

where the so called Kraus generators of that map are

$$S_0(t) = \begin{pmatrix} \cos(\lambda t) & 0 \\ 0 & 1 \end{pmatrix}, \quad S_1(t) = \begin{pmatrix} 0 & 0 \\ \sin(\lambda t) & 0 \end{pmatrix}, \quad (20)$$

and satisfy the property $S_0^\dagger S_0 + S_1^\dagger S_1 = \mathbf{1}$. Since in this case no EM field is in the cavity, ε_U corresponds to spontaneous decay of the excited atom state (as opposed to stimulated decay if $|f\rangle$ is not $|0\rangle$), through the emission of a photon.

Suppose now that we have ε_{V^2} evolution with $U = e^{i\frac{\pi}{4}\sigma_2}$, and that the coin state is not a classical mixture of states but rather that the coin starts up in the state $|c\rangle = \cos \chi |+\rangle + i \sin \chi |-\rangle$, $\rho_C = |c\rangle \langle c|$, then eq. (11) tells us that

$$\rho_C^1 = \frac{1}{2} \mathbf{1} + \frac{1}{2} \sqrt{1-\gamma} \sin 2\chi \sigma_2 + \frac{1}{2} [(1-2\gamma) \cos^2 \chi - \sin^2 \chi] \sigma_3. \quad (21)$$

Since in this case $A(\chi, \gamma) = -[(1 - 2\gamma) \cos^2 \chi - \sin^2 \chi]$ and $B(\chi, \gamma) = \sqrt{1 - \gamma} \sin 2\chi$, we have that $C(\chi, \gamma) = \sqrt{[(1 - \gamma) \cos 2\chi - \gamma]^2 + (1 - \gamma) \sin^2 2\chi}$, and the asymptotic characteristic function is $h(\phi; \chi, \gamma) = A(\chi, \gamma) \cos 2\phi + B(\chi, \gamma) \sin 2\phi$. This gives us that the mean position is zero and that the standard deviation is $\sigma(\chi, \gamma) = \frac{C(\chi, \gamma)}{\sqrt{2}}$. Note that even in the case of very strong decay ($\gamma = 1$) the asymptotic mean position of the walker remains zero. This is due to the symmetrizing effect of U . However the standard deviation of the position depends explicitly on the decay rate. Nevertheless it is not monotonic in the decay rate. Since $\sigma^2(\chi, \gamma) = \frac{1}{2}[1 - 4\gamma(1 - \gamma) \cos^4 \chi]$, when $\gamma = 0$ or 1 , we have $\sigma^2 = \frac{1}{2}$, and this is maximum spread. Furthermore the spread decreases until $\gamma = \frac{1}{2}$ and then increases again. So we can say that the effect of spontaneous emission is to *decrease* the spread of the walker position distribution, however the minimum spread is achieved when the probability of decay is $\frac{1}{2}$.

V. CONCLUSIONS

Here we have analysed the asymptotic behaviour of the ε_{V^*} quantum random walk on \mathbb{Z} . Firstly, we see that generically the standard deviation of the position distribution increases linearly with the number of steps, which is a quadratic speedup over the classical random walk. However there are resonant quantum random walks in which the asymptotic characteristic function $h(\phi)$ is constant, which spread classically. Off resonance the normalized position random variables $\frac{L_c^{(n)}}{n}$, are shown to converge weakly to the random variable $Y = h(\phi)$, where ϕ , is a random variable whose distribution depends on the initial *walker* density matrix. Furthermore, the asymptotic distribution of the ε_{V^2} model, off resonance is obtained and it is found that if the position random variable is normalized, then the asymptotic distribution assumes a universal form given in theorem 4. Secondly, it is shown how to drive the asymptotic behaviour of the ε_{V^2} quantum random walk through the use of a Jaynes-Cummings model interaction that prepares the coin system. It is found that by a judicious choice of the initial coin state, it is possible to drive the quantum asymptotic behavior of the walk to classicality by tuning the time spent by coin to cross the Jaynes-Cummings cavity. Finally, the effect of spontaneous emission of the coin is considered and it is found that in the particular case of ε_{V^2} evolution with $U = e^{i\frac{\pi}{4}\sigma^2}$, it preserves the symmetry of the walk, and it actually decreases the spread in the walker position.

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